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Cutoff frequencies of a clarinet. What is acoustical regularity?

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Summary
A characteristic of woodwind instruments is the cutoff frequency of their tone-hole lattice. Benade proposed a practical definition using the measurement of the input impedance, for which at least two frequency bands appear. The first one is a stop band, while the second is a pass band. The value of this cutoff frequency, which is a global quantity, depends on the whole geometry of the instrument, but is rather independent of the fingering. This seems to justify the consideration of a woodwind with several open holes as a periodic lattice. However the holes on a clarinet are very irregular. The paper investigates first the experimental method of determination of the cutoff frequency, then the question of the acoustical regularity: an acoustically regular lattice of tone holes is defined as a lattice built with T-shaped cells of equal eigenfrequencies. Then the paper discusses the possibility of division of a real lattice into cells of equal eigenfrequencies. It is shown that it is not straightforward but possible, explaining the apparent paradox of the Benade theory. From this division, a narrow range of possible constant cutoff frequencies is found, and it is in very good accordance with the experimental results from impedance measurements. When considering the open holes from the input of the instrument to its output, the distances between holes are enlarged together with their radii: this explains the relative constancy of the eigenfrequencies.

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1. Introduction

In 1960 Benade [1] has the idea to use the theory of periodic media in order to understand the properties of woodwind instruments, provided with lattices of tone holes. The existence of a cutoff frequency was discovered, and he showed that it is easy to measure. He measured many instruments, either modern or historical, either cylindrical or conical, and obtained the result that the cutoff frequency due to open holes does not vary much with the fingering, i.e. with the number of tone holes [2]. This seems to justify the use of the theory of periodic media.

Nevertheless for the main holes of a clarinet, the radius varies between 2.5mm and 5.7mm, and the spacing between typically 10mm to 30mm. This means that geometrical regularity does not exist for a real lattice.

In this paper, we investigate some questions:

- How to measure the cutoff for a given fingering?
- Is it possible to define an acoustical regularity instead of a geometrical one?
- Is a clarinet an acoustically regular lattice of open tone holes?

The two last questions have been discussed in detail in a paper recently submitted [3].

2. Experimental determination of the cutoff frequency

Benade [2] proposed to determine the cutoff frequency from the measurement of the input impedance curve. Above a certain frequency, the extrema of the impedance modulus become less pronounced (see Fig. 1), because of a great increase of the radiation damping, all holes contributing to radiation. In general a change in slope can be observed. This definition is not perfect, but it is efficient in practice.
We will first examine what happens for ideal cases. We recall that three conditions need to be fulfilled for an unambiguous definition of cutoff frequencies:

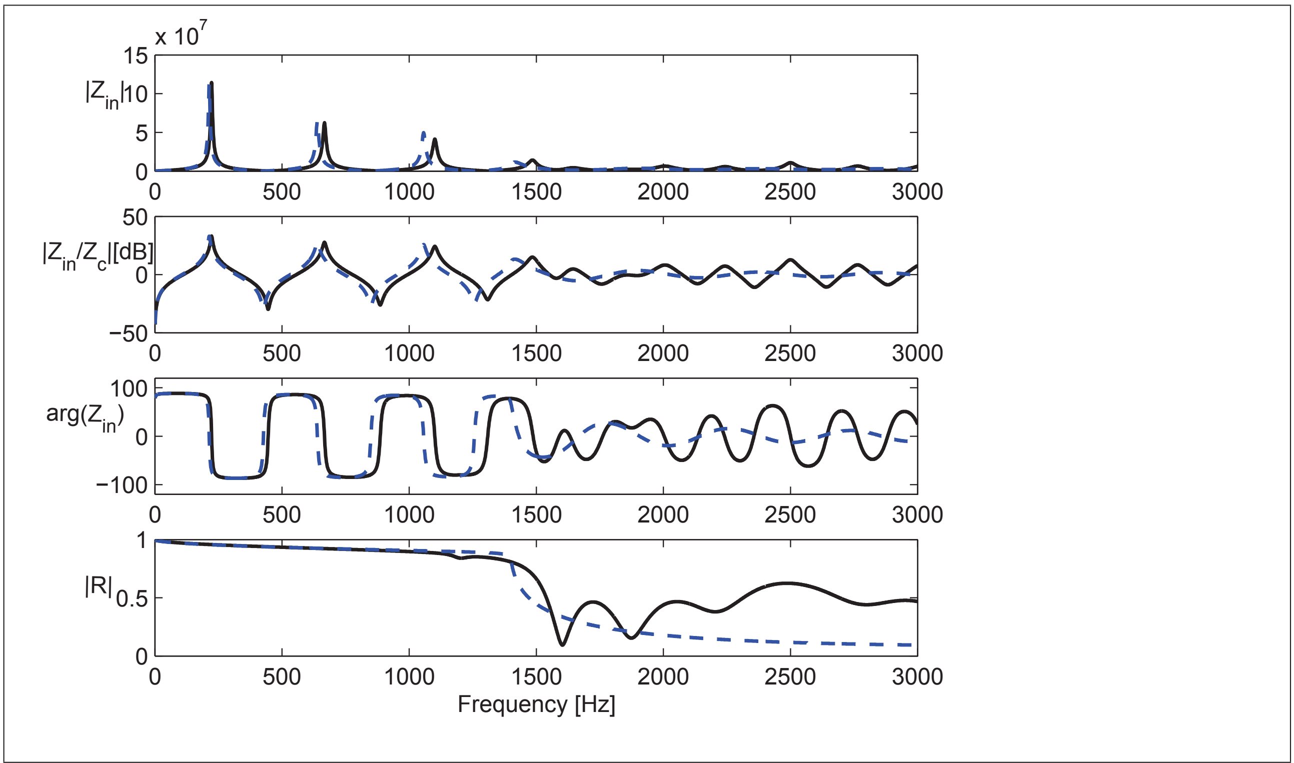

Figure 1. Input impedance of a modified clarinet with 6 open holes. Solid line: irregular lattice replaced with an infinite regular lattice of open tone holes. Dotted line: irregular lattice replaced by a regular lattice of same length. From top to bottom: modulus of the impedance (linear scale): modulus of the impedance (logarithmic scale); argument of the impedance; modulus of the reflection coefficient

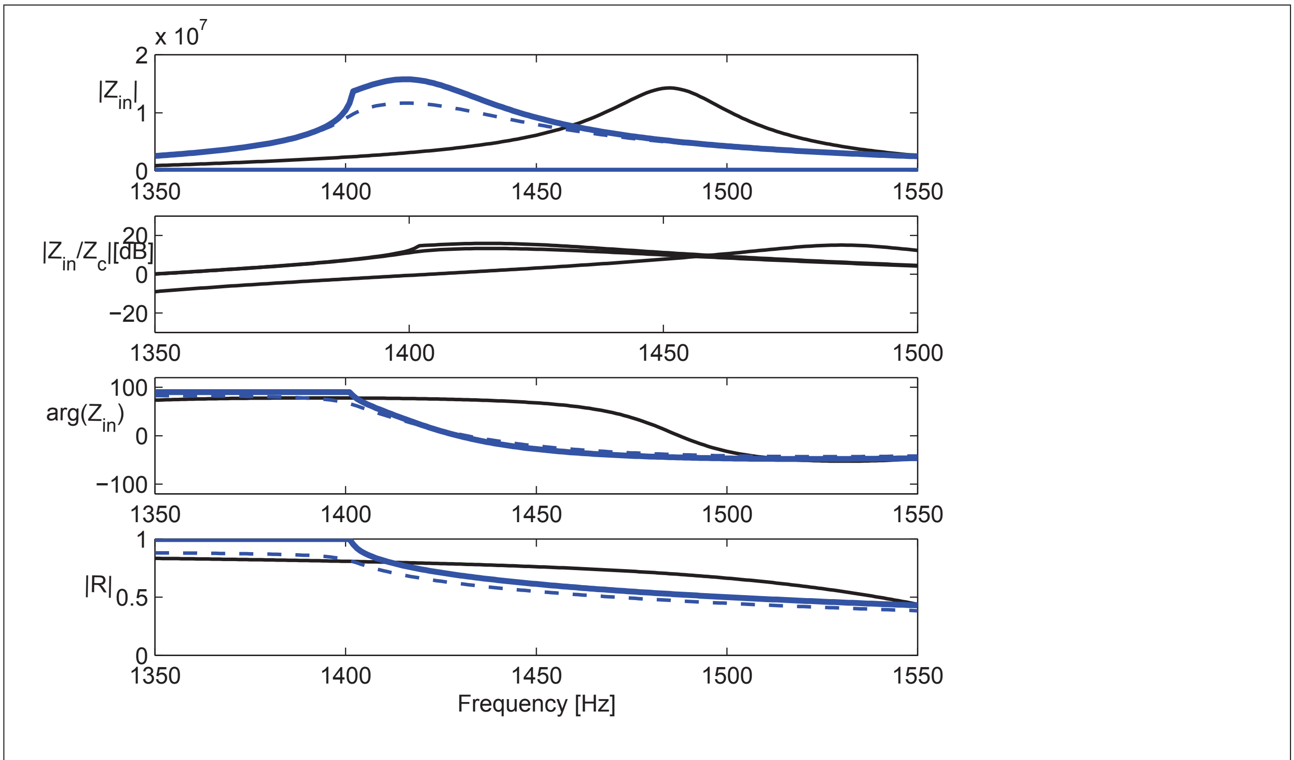

Figure 2. Zoom of Fig. 1 around the cutoff (1402 Hz). A thick line has been added for the case of an infinite, lossless lattice

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i) The lattice is lossless. Without losses, the propagation constant is real in stop bands (evanescent waves) and purely imaginary in pass bands (propagating waves). When losses exist, waves are always both propagating and attenuated, i.e. the propagation constant is always complex. Thus the cutoff does not exist rigorously. Actually this limitation is rather weak in practice, because the distinction between two bands remains clear for small losses.

ii) The lattice is infinite. Waves are propagating or exponentially decreasing in one sense only. For a finite lattice, reflections occur at the termination, and the separation between stop and pass bands in not clear. Near the termination, an exponentially increasing wave exists when the propagation constant is real, and can even allow power transmission (by interaction with the decreasing wave - this is the well known tunneling effect of quantum mechanics). Notice that for time functions, periodicity is strictly defined for functions of infinite extent only, leading to possible expansion in Fourier series. This strict definition of periodicity could be used also in space for lattices, and makes this second condition redundant with the following one.

iii) The lattice is periodic. The Floquet (or Bloch) theory has been elaborated for periodic media. For irregular or even random media, the distinction between stop and pass band does not exist (at the limit of random media of infinite extension, only stop bands exist).

It is interesting to examine how Benade's method works for a regular medium. If the lattice is infinite and regular, the input impedance of the lattice is its characteristic impedance, and if it is lossless, this impedance is either purely imaginary in stop bands or real in pass bands. As a consequence the separation is clear, and it will be more evident on the argument of this impedance, exhibiting a jump of $\pm\pi/2$. This observation can be different at the input of the instrument, upstream to the lattice (i.e. at the input of a tube of a certain length ℓ terminated in the lattice of open holes). Figure 2 shows that a kink exists on all curves of the input-impedance argument of the instrument when losses are ignored. When losses are taken into account, the kinks disappear, but the behavior remains similar, and it is possible to distinguish two bands for the argument of the input impedance. However it is possible to find other useful quantities. The reflection coefficient of planar waves at the input of the lattice is directly related to the input impedance of the lattice, and does not depend on the tube upstream of the lattice. This is true at least if the tube is perfectly cylindrical, as supposed for the example shown on Figs. 1 and 2. The cutoff clearly appears for the infinite lattice, even with losses.

What happens for a finite, regular lattice? The same representation shows the interest of the modulus of the reflection coefficient. However the determination of the cutoff is not clear. While for the infinite lattice the cutoff is 1402 Hz, the limit between the two first bands seems to be higher than 1500 Hz for the finite one. The regular lattice considered has 6 open holes (it has been obtained from the real one by equalization of the cells see Ref. [3]).

What happens for a real lattice, i.e. a lossy, irregular and finite lattice? Figures 3 and 4 show that a change in slope clearly appears for all representations (the linear scale for $|Z_{in}|$ is not kept because it is less appropriate), the modulus of the reflection coefficient being the most appropriate. There is a paradox: the separation between the two first bands is more clear for an irregular lattice than for a regular one. This seems to be true for almost all fingerings of a clarinet. We have no interpretation for that.

3. Variation of the cutoff frequency with the fingering

If it is assumed that for a real lattice, the cutoff is given by a kink in the input impedance curve, it is remarkable that the measured cutoff varies very little with the fingering, i.e. with the number of open holes. On this point of view, a clarinet seems to have the behavior of a regular lattice. As stated in the introduction, this seems to justify the theory of Benade, based on the theory of periodic media. This property seems to be rather general.

Figure 5 shows the result for the first register of a clarinet. The cutoff is rather constant over the major part of the register, from A♯3 to A4, around 1450 Hz. For the lowest notes, the measured cutoff is around 1200 Hz. A first interpretation could be done as follows: for each hole, it is possible to define a "local" cutoff, i.e. the cutoff of the periodic lattice having same hole dimensions and spacing, therefore it seems that the holes are arranged into two groups with two different values of the local cutoff. This interpretation is not correct, because:

i) The definition of a local cutoff is far from evident. In principle it should be the cutoff of a periodic lattice built as a repetition of a given cell. The difficulty lies in the division of the real lattice in cells. How to choose the lengths of the cells? This question is discussed in detail in Ref. [3].

ii) When the real lattice is replaced by a regular lattice of same length and same number of open holes, it is possible to find a cutoff close to the average cutoff for the upper notes of the first register (see Fig. 1), but a similar result is found for the lower notes. For these notes the cutoff is found to be different from the theoretical one (for a regular lattice it is the local one), and close to 1200 Hz. As a conclusion, the existence of a low cutoff for the lower notes is an effect of termination (i.e. the lower part of the pipe, including

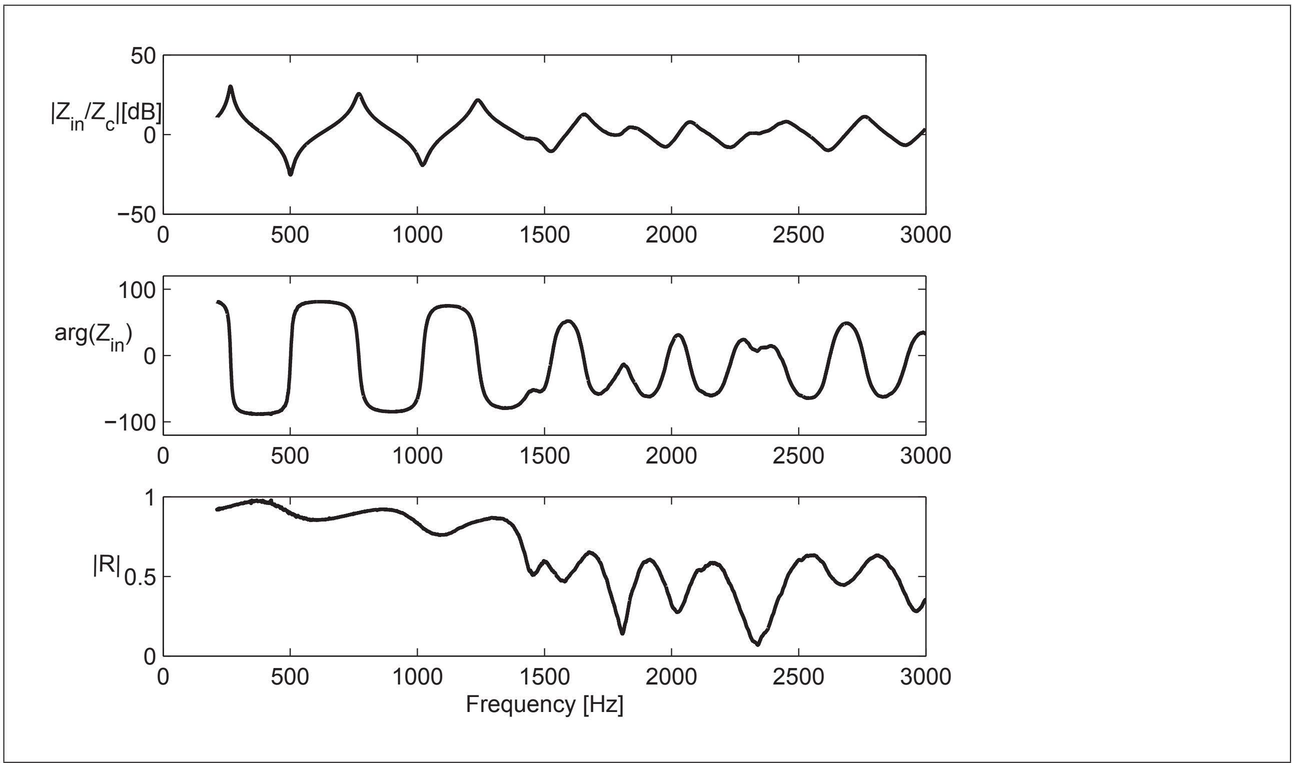

Figure 3. Input impedance (modulus and argument), and modulus of the reflection coefficient measured for the fingering D4

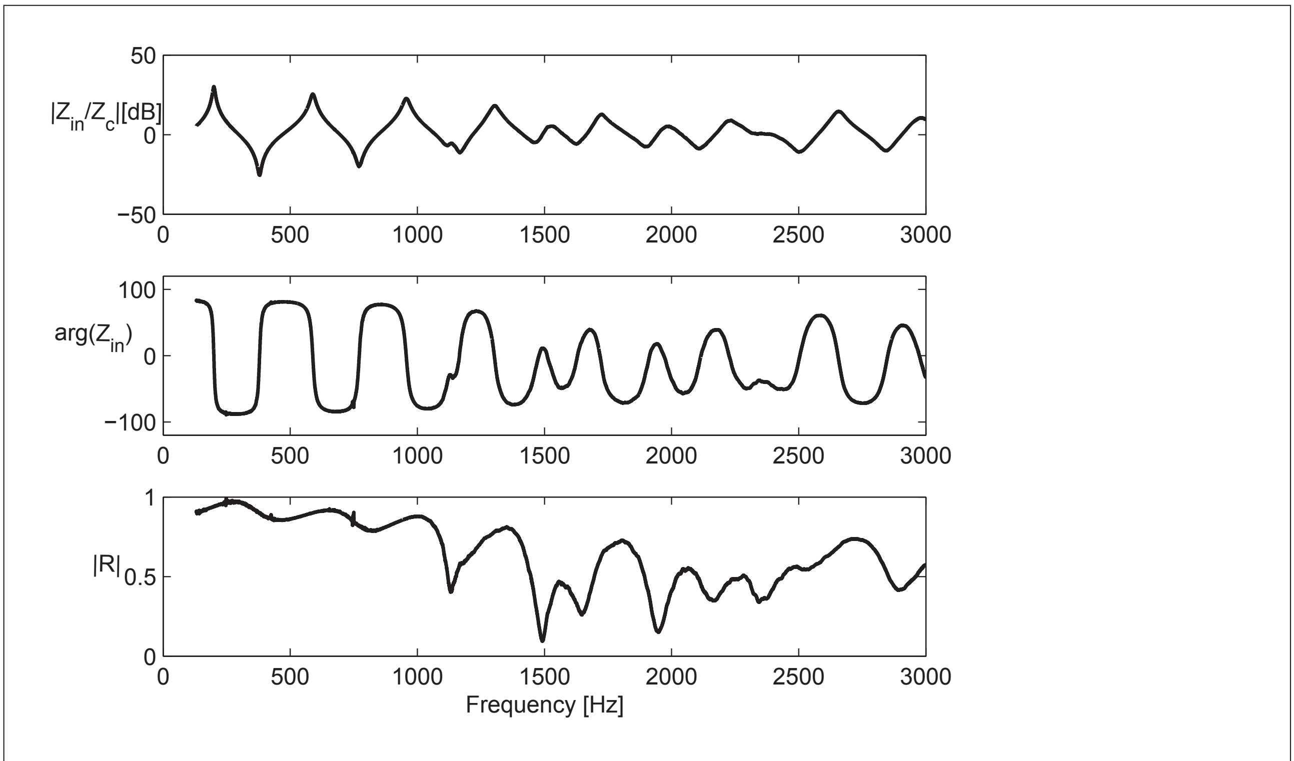

Figure 4. Input impedance (modulus and argument), and modulus of the reflection coefficient measured for the fingering A3

the bell). This effect appears for both the real lattice and the regular one: when modifying the geometry of the lower part of the instrument, the cutoff frequency can be strongly modified.

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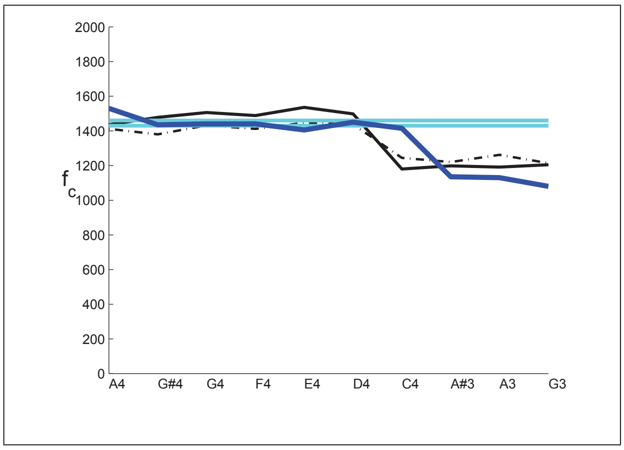

Figure 5. Cutoff frequencies for the first register of a clarinet. Thick, solid line: measured frequencies; thin, solid line: calculated frequencies for the model of acoustically regular lattice; dash-dot line: calculated frequency for a geometrically regular lattice; horizontal, pale lines: limit of common eigenfrequencies for the division into Helmholtz resonators

4. What is an acoustically regular lattice?

One question remains: why the property of a real, irregular lattice seems to be so similar to this of a regular one? A rapid inspection of a clarinet shows that the holes are very different, with very different spacing. An answer to the question has been found by defining the concept of acoustical regularity. It is actually possible to extend the Floquet theory to a certain kind of irregular media. Let us consider a succession of two-port networks, represented by a product of transfer matrices, as follows:

$$\prod_i^N \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$$

The transfer matrices, relating pressure and flow rate vectors at the input and the output of a two-port, are assumed to be unitary (because reciprocity) and symmetrical ($A_i = D_i$).Therefore they can be equivalently described by the two following quantities: the propagation constant Γ_i, such as $\cosh\Gamma_i = A_i$ and the characteristic impedance $\mathcal{Z}_{ci} = \sqrt{B_i/C_i}$. An interesting case is this of matrices having the same $\mathcal{Z}_{ci} = \mathcal{Z}_c$ (see [4]). The product of the matrices is easy to calculate:

$$\prod_i \begin{pmatrix} \cosh\Gamma_i & \mathcal{Z}_c \sinh\Gamma_i \\ \mathcal{Z}_c^{-1}\sinh\Gamma_i & \cosh\Gamma_i \end{pmatrix} = \begin{pmatrix} \cosh\sigma & \mathcal{Z}_c \sinh\sigma \\ \mathcal{Z}_c^{-1}\sinh\sigma & \cosh\sigma \end{pmatrix}$$

where $\sigma = \sum_i \Gamma_i$. The behavior is similar to that of the regular medium with the same total propagation constant $\sigma = n\Gamma$. As a consequence, if a lattice is built with irregular cells having the same characteristic impedance *at every frequency*, its behavior is the same as the behavior of a perfectly periodic medium. If this situation can exist for wind instruments, acoustical regularity can exist without geometrical regularity. In particular stop and pass bands can exist: when σ is imaginary (and $\mathcal{Z}_c$ real), waves propagate, while if σ is real (and $\mathcal{Z}_c$ imaginary), waves are evanescent.

An examination of the cells of a lattice built with a cylindrical tube with open tone holes leads to the conclusion that acoustical regularity can exist for frequencies with wavelengths larger than the dimensions of a cell if:

- the dimensions of the cells are smaller than the wavelength at the cutoff frequency;
- the cutoff frequencies of the cells are identical.

For a symmetrical cell, it can be proven that whatever the dimensions, the first cutoff frequency is exactly the eigenfrequency of a cell closed at the two extremities (see Fig. 6). Therefore if symmetrical cells having same eigenfrequencies are connected in cascade, an acoustically regular lattice is obtained, at least at low frequencies.

Notice that a cell of the lattice for the case of wind instruments is nothing else than a Helmholtz resonator. This means that even if the neck of the resonator is not at the middle (the cell being not symmetrical), the eigenfrequency is the same as this of a symmetrical cell with the same neck, assuming the condition concerning the wavelength to be valid.

The conclusion is: an acoustically regular lattice can be built by juxtaposing Helmholtz resonators with the same eigenfrequency. The condition is that dimensions are smaller than the wavelength at the cutoff frequency.

5. Acoustical regularity of a clarinet

Is it possible to regard a clarinet as a lattice of this kind? A first rough indication is that when considering holes at increasing distances from the reed, they seem to be both more spaced and larger. Increasing distance can compensate for increasing radius in order to get constant eigenfrequency. This is confirmed by a precise examination: we do not discuss here the method of analysis of a real lattice, i.e. the division of the lattice into Helmholtz resonators, but we have found that such a division is possible. It is not unique, the possible (common) cutoff frequency lying between 1430 Hz and 1460 Hz, very close to the experimental value obtained for the higher part of the first register.

6. Conclusion

Up to 1500 Hz, a clarinet can be seen as an acoustically regular lattice, i.e. the variation in radius of the holes is offset by the variation in spacing, resulting in the existence of resonators of similar eigenfrequencies.

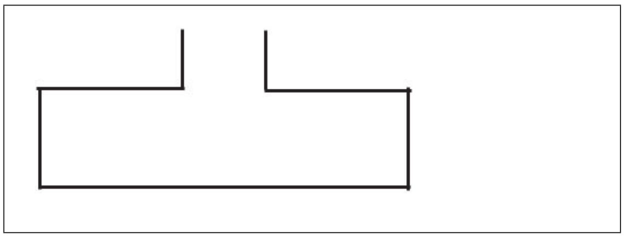

Figure 6. A cell of lattice of tone hole viewed as a Helmholtz resonator

The measured cutoff frequency is rather constant, this effect being this of acoustical regularity, not of geometrical regularity. The variation of measured cutoff for lower notes is due to a termination effect, not of irregularity.

Several questions remain: what happens slightly above cutoff? Why cutoff seems to be easier to measure for a real lattice than for a purely regular one? Is a constant value for the cutoff over the different fingerings a goal for the makers? For the latter question, we refer to Benade [5].

The question of the effect of the cutoff value on the produced sound is out the scope of the paper (see Refs. [6, 7]).

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